MOP Exercises

1. Recall the definition of quadratic variation of an L^2-martingale, (X_t):

$$\langle X \rangle_t = \lim_{\theta} \sum_{\theta} \Delta X_{t_i}^2$$

for partitions θ of $[0, t]$. Assume that the limit exists in probability. Show that,

(i) If $0 \leq s \leq t$ then $0 \leq \langle X \rangle_s \leq \langle X \rangle_t$.

(ii) For $t \geq 0$, $\langle X \rangle_t$ is $\mathcal{F}_t$-measurable.

(iii) For a simple process, $f(s) = \sum_i f_{t_{i-1}} I_{(t_{i-1}, t_i]}(s)$, where $f_{t_{i-1}}$ is $\mathcal{F}_{t_{i-1}}$ measurable, prove the isometry property:

$$\left\| \int_0^t f(s) \, dX_s \right\|_2^2 = \mathbb{E}\left(\int_0^t |f_s|^2 \, d\langle X \rangle_s \right).$$

State your assumptions about the simple process f.

2. Let $X = (X_t)$, $Y = (Y_t)$ be L^2 martingales. Observe that,

$$X_t Y_t = \frac{1}{4}\left((X_t + Y_t)^2 - (X_t - Y_t)^2 \right)$$

We know from Theorem 0.1 that each of

$$(X_t + Y_t)^2 - \langle X + Y \rangle_t \quad , \quad (X_t - Y_t)^2 - \langle X - Y \rangle_t$$

are L^1-martingales. Deduce that

$$X_tY_t - \frac{1}{4}\left(\langle X+Y\rangle_t - \langle X-Y\rangle_t\right)$$

is an L^1 martingale. Note that

$$\langle X,Y\rangle_t = \frac{1}{4}\left(\langle X+Y\rangle_t - \langle X-Y\rangle_t\right)$$

So we have proved that $(XY - \langle X,Y\rangle)$ is an L^1 martingale. Since $\langle X+Y\rangle$ and $\langle X-Y\rangle$ are bounded variation natural processes, so is $\langle X,Y\rangle$. Also, a generalisation of Theorem 0.1 holds: $\langle X,Y\rangle$ is the unique natural bounded variation process such that $(XY - \langle X,Y\rangle)$ is an L^1-martingale. Use this to prove

(i) $\langle X, X\rangle = \langle X\rangle$

(ii) $\langle X, Y+Z\rangle = \langle X,Y\rangle + \langle X,Z\rangle$

(iii) $\langle X, Y\rangle = \langle Y, X\rangle$

(iv) $\langle \lambda X, Y\rangle = \lambda\langle X,Y\rangle$ for $\lambda \in \mathbb{R}$.

3. A stopping time is a map, $\tau : \Omega \to [0,\infty]$ (time) such that $\{\tau \leq t\} \in \mathcal{F}_t$. This last condition can be paraphrased; regarding τ as the (random) time of occurence of some set of (random) circumstances and regarding $\mathcal{F}_t$ as the information known at time t, the

condition $\{\tau \leq t\} \in \mathcal{F}_t$ is simply that given a state of the world, $\omega \in \Omega$, one 'knows' whether or not τ has 'occured' by time t, that is $\omega \in \{\tau \leq t\}$ or $\omega \notin \{\tau \leq t\}$.

An example a stopping time is a "first arrival time". Let $L > 0$ and (W_t) a standard BM. Define $\tau(\omega) = \inf\{t : W_t(\omega) \geq L\}$. Since the paths of BM are continuous this is "the first time W hits L" and is its first arrival in the set $[L, \infty)$. Proving that this kind of time is a stopping time is not always easy. See below.

α Let σ and τ be stopping times. Prove that $\sigma \vee \tau = \max\{\sigma(\omega), \tau(\omega)\}$, $\sigma \wedge \tau = \min\{\sigma(\omega), \tau(\omega)\}$ are stopping times.

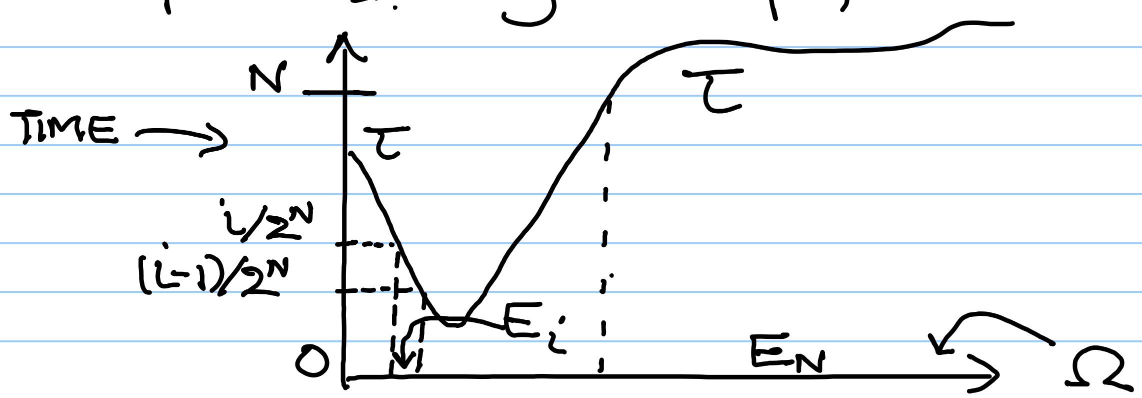
β Let $t \in [0, \infty]$. Prove that $\hat{t}(\omega) = t$ is a stopping time.

γ Let σ be a stopping time and $t > 0$. Is $\sigma + t$ a stopping time? Details please.

δ Let (σ_n) be a sequence of stopping times. If $\sigma_1 \leq \sigma_2 \leq \sigma_3 \ldots$ and $\sigma_n(\omega) \uparrow \sigma(\omega)$ for each $\omega \in \Omega$, is σ a stopping time? If $\sigma_1 \geq \sigma_2 \geq \ldots$ and $\sigma_n(\omega) \downarrow \sigma(\omega)$ is σ a stopping time? Details please.

ε Let $\tau : \Omega \to [0, \infty]$ be a stopping time. Let $N \in \mathbb{N}$ and partition Ω as follows. Let $E_N = \{\tau > N\}$ and for $1 \leq i \leq N2^N$ let $E_i = \{\omega : \frac{i-1}{2^N} < \tau \leq \frac{i}{2^N}\}$, and $E^0 = \{\tau = 0\}$.

A picture might help;



Define $\tau_N(\omega) = \begin{cases} 0 \text{ on } E^0 \\ \infty \text{ on } E_N \\ i/2^N \text{ on } E_i \end{cases}$

If $\omega \in \Omega$ and $\tau(\omega) = \infty$ then $\tau_N(\omega) = \infty$.
If $\omega \in \Omega$ and $\tau(\omega) < \infty$ then there is an $N \in \mathbb{N}$ such that $\tau(\omega) < N$ and $|\tau_M(\omega) - \tau(\omega)| < \frac{1}{2^M}$ for $M \geq N$. Show that (τ_N) decreases with N and converges pointwise to τ. Show that τ_N is a stopping time.

4. (i) Let $\tau(\omega) = t\, I_E(\omega) + \infty\, I_{\Omega \setminus E}(\omega)$ where $E \in \mathcal{F}_t$. Prove that τ is a stopping time.

(ii) Let $X = (X_t)$ be an L^1 process. Write down X_τ explicitly.

(iii) If $X = (X_t)$ is an L^1 martingale of the form $X_t = M_t(X_\infty)$ prove that $\mathbb{E}(X_\tau) = \mathbb{E}(X_\infty)$ (no matter which t or E we choose).

(iv) Let X be an L^1 process such that $\mathbb{E}(X_\tau)$ is a constant for all bounded stopping times. By considering the stopping times $\hat{t}$ and $\tau(\omega) = s I_E + t I_{\Omega \setminus E}$ for $s < t$ and $E \in \mathcal{F}_s$, prove that X is an L^1 martingale.

5. Let σ be a stopping time taking a finite number of values, $t_0, t_1, \ldots, t_n$. Let X be an L^1 martingale. Then

$$X_\tau = \sum_{i=0}^{n} X_{t_i} I_{E_i}, \quad E_i = \{\tau = t_i\}.$$

(assume $t_0 < t_1 < t_2 \ldots$ etc).

Prove that if $t \le t_n$

$$X_{\tau\wedge t} = \sum_{t_{i-1}\le t} X_{t_i} I_{E_i} + X_t I_{\{\tau \geq t\}}$$

Prove that the process $(X_{\tau\wedge t})$, $t \ge 0$, is an L^1-martingale.

6. Let $X = (X_t)$ and $Y = (Y_t)$ be independent Brownian Motions. Prove that $\langle X, Y\rangle_t = 0$. Hint: Look at the limit over partitions of $[0,t]$ of $\sum_{\pi} \Delta X_{t_i} \Delta Y_{t_i}$.

7. Let $(\Omega, \mathcal{F}_2, \mathbb{P})$ be a probability space and $\mathcal{F}_1 \subseteq \mathcal{F}_2$ a sub-σ-field of $\mathcal{F}_2$. Let M be the conditional expectation of $L^2(\Omega, \mathcal{F}_2, \mathbb{P})$ onto $L^2(\Omega, \mathcal{F}_1, \mathbb{P})$. This is a filtration, albeit with only two time points.

Prove that any stopping time of this filtration, i.e. a map $\tau : \Omega \to \{1,2\}$ with $\{\tau \le t\} \in \mathcal{F}_t$, for $t \in \{1,2\}$, must have the form,

$$\tau(\omega) = I_E(\omega) + 2 I_{\Omega\setminus E}(\omega) \qquad E \in \mathcal{F}_1.$$

Note that this says that there are as many 'distinct' stopping times as there are 'distinct' sets in $\mathcal{F}_1$ despite there only being two time points. Let (X_1, X_2) be a process adapted to this filtration. We are interested in finding a time, τ^*, such that

$$\mathbb{E}(X_{\tau^*}) = \sup_{\tau} \mathbb{E}(X_\tau)$$

Show that there is a set in $\mathcal{F}_2$, E say, such that

$$\mathbb{E}(X_1 I_E + X_2 I_{\Omega \setminus E}) \geq \sup_{\tau} \mathbb{E}(X_\tau)$$

and identify this set explicitly (in terms of X_1 and X_2). Why can't we just use E to make the stopping time τ^* immediately? Next, observe, for any $F \in \mathcal{F}_2$

$$\mathbb{E}(X_1 I_F + X_2 I_{\Omega \setminus F}) = \mathbb{E}(M(X_1 I_F + X_2 I_{\Omega \setminus F})).$$

Prove that if $\tau = X_1 I_F + X_2 I_{\Omega \setminus F}$, $F \in \mathcal{F}_1$

then $\mathbb{E}(X_\tau) = \mathbb{E}(X_1 I_F + M(X_2) I_{\Omega \setminus F})$.

Revisit that set, E, identified above, and use the insight obtained there to identify an $F \in \mathcal{F}_1$ that allows you to construct τ^*.

For questions 8, 9, 10, refer to the hand-out distributed in our first lecture.

Exercises 2

1. Prove the integration by parts formula for semimartingales X, Y.

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s \, dY_s + \int_0^t Y_s \, dX_s + \langle X, Y \rangle_t .$$

Hint: $f(x,y) = xy$.

2. Suppose that

$$X_t = X_0 + \int_0^t \mu_X X_s \, ds + \int_0^t \sigma_X X_s \, dW_s^X$$

with Y_t "identical". Use question 1 to show that

$$X_t Y_t = X_0 Y_0 + \int_0^t \mu_X X_s Y_s \, ds + \int_0^t \sigma_X X_s Y_s \, dW_s^X + \int_0^t \mu_Y X_s Y_s \, ds$$
$$+ \int_0^t \sigma_Y X_s Y_s \, dW_s^Y + \int_0^t \sigma_X \sigma_Y XY \, d\langle W^X, W^Y \rangle_s .$$

Explain how you got to this formula.

3. A supermartingale is a process, (X_t), for which

$$M_s(X_t) \leq X_s , \qquad s < t ,$$

while a submartingale satisfys the reverse inequality.

$$M_s(X_t) \geq X_s .$$

Look at the properties of conditional expectations given to you in Stochastic Processes (you are looking for a "Schwartz Inequality"). Use this to prove that if X is an L^2 martingale then X^2 is a submartingale.

4. Let τ be a time taking only finitely many values, $t_0, t_1, \ldots t_n$. Then for any process, (X_t),

$$X_\tau = \sum_{i=0}^{n} X_{t_i} I_{\{\tau = i\}} .$$

Suppose now that X is a supermartingale. Prove that the process, $(X_{\tau \wedge t})$, is a supermartingale too.

5. Let f and g be simple processes each of the form,

$$h(s) = \sum_{i=1}^{k} h_{t_{i-1}} I_{[t_{i-1}, t_i)}(s)$$

(i.e. you get $f(s)$ and $g(s)$ just by replacing h with f or g respectively).
Let X and Y be L^2- martingales. Prove that,

$$Z_t = \left(\int_0^t f(s)\, dX_s \right)\left(\int_0^t g(s)\, dY_s \right) - \int_0^t f(s) g(s)\, d\langle X, Y \rangle_s$$

is an L^1- martingale.

6. Recall the definition of the correlation coefficient for random variables, X, Y.

$$\rho_{XY} = \frac{\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)}{\sqrt{var(X)}\sqrt{var(Y)}}$$

$$\equiv \frac{Cov(X, Y)}{\sigma_X \; \sigma_Y}$$

here $Cov(X,Y)$ is the co-variance of X and Y and $\sigma_X = \sqrt{Var(X)}$ is sometimes called the standard deviation of X. By the way, $Cov(X,Y)$ is (yet) another of these bilinear forms arising from a quadratic form — $Var(X)$..... .

Let W^1 and W^2 be Brownian Motions. Recall the fundamental result, $\langle W^1, W^2 \rangle$ is the unique........ such that $W^1W^2 - \langle W^1, W^2 \rangle$ is a martingale. Use this fact to deduce that,

$$\rho_{W^1_t, W^2_t} = \frac{\mathbb{E}(\langle W^1, W^2 \rangle_t)}{t} .$$

7. Let W^1, W^2 be independent Brownian motions. Define,

$$W^3 = \rho W^1 + (1-\rho^2)^{1/2} W^2$$

i.e. $$W^3_t = \rho W^1_t + (1-\rho^2)^{1/2} W^2_t$$

where ρ is a number in $[-1,+1]$. Show that $\langle W^1, W^3 \rangle = \rho \langle W^1, W^1 \rangle$ and that $\rho_{W^1_t W^3_t} = \rho$.

When we have proved Levy's Theorem: prove that W^3 is a Brownian Motion.

8. Let $S_t = S_0 + \int_0^t \mu^S S_s \, ds + \int_0^t \sigma^S S_s \, dW^S_s$ where

S_0, μ^S and σ^S are positive constants. The process (S_t) is the price of a foreign asset in a foreign economy viewed from the domestic economy. So S could represent the \$ price of Berkshire Hathaway over time. Let (f_t) be the exchange rate £/\$ so that 1\$ is £f_t at time t. Then $(S_t f_t)$ represents the price process of BRK.A in £'s sterling. Let

$$f_t = f_0 + \int_0^t f_s \gamma^f ds + \int_0^t f_s \sigma^f dW_s^f$$

where f_0, γ^f, σ^f are positive constants. The Brownian motions, W^S, W^f have constant correlation, $\rho^{(\dagger)}$. Write down a stochastic equation for $(f_t S_t)$.

($\dagger$) I mean something formally stronger:

$$\langle W^S, W^f \rangle_t = \rho t \quad \forall t.$$

9. Let $X^n \equiv (X_t^n)$ be a sequence of martingales in $\mathcal{M}^2$. This means that for each n, $X_t^n = M_t(X_T^n)$ (T is the "end of time" ....), for each $t \leq T$. Recall the correspondence between $\mathcal{M}^2$ and $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$:

$$\mathcal{M}^2 \ni (X_t) \equiv (M_t(X_T)) \longleftrightarrow X_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$$

Recall also that $\mathcal{M}^2$ "inherits" the structure from $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$; in particular the norm,

$$\|(X_t)\| = \|X_T\|_2$$

This means that a sequence (X^n) in $\mathcal{M}^2$ converges to X, in $\mathcal{M}^2$, iff

$$\|(X_t^n) - (X_t)\|_2 = \|X_T^n - X_T\|_2 \to 0 \text{ as}$$

$n \to \infty$. But something a little deeper is actually going on here: Suppose that for each n, $X^n \in \mathcal{M}^2$ and for each $t \leqslant T$, $X^n_t \xrightarrow{L^2} X_t$. Then,

① (X_t) is an adapted process

② (X_t) is a martingale on $[0,T)$ (note 'T,' not in)

③ $(X_t) \in \mathcal{M}^2$.

Prove each of these assertions; some hints.

① For a fixed t, for each n, $X^n_t \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$.

② $M_s(X_t) = M_s(\lim_n X^n_t) = \lim_n M_s(X^n_t)$

③ Because $X^n_T \xrightarrow{L^2} X_T$ this means that $X^n \to X$ in $\mathcal{M}^2$. In fact, (X^n) is a Cauchy sequence in $\mathcal{M}^2$ iff (X^n_T) is a Cauchy sequence in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. But $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ is complete, so there is an $X_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ such that $X^n_T \to X_T$. It follows that $X^n \to X \equiv (M_t(X_T))$ in $\mathcal{M}^2$, i.e. $\mathcal{M}^2$ is complete.....

If we had not included $t = T$ in the condition "$X^n_t \xrightarrow{L^2} X_t$" would ③ remain true? I mean, suppose $X^n \in \mathcal{M}^2$ (so every $X^n \equiv (M_t(X^n_T))$) and for $t < T$, $X^n_t \to X_t$. Would $(X_t) \in \mathcal{M}^2$? Think about $T = \infty$ and (X_t) being a Brownian Motion. Is Brownian Motion a martingale of the form $(M_t(X_\infty))$ for $X_\infty \in L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$? Come to that, for (W_t) a BM, the random variable $W_t \equiv N(0,t)$, i.e.

$$\mathbb{P}(W_t \in H) = \int_H \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \qquad H \subseteq \mathcal{B}(\mathbb{R}).$$

What happens as $t \to \infty$? Let $\hat{n} = n I$ be the stopping time, "n". Think about??

$$W^{\hat{n}} \equiv (W_{\hat{n} \wedge t}).$$

Is $W^{\hat{n}} \in \mathcal{M}^2$? What is $\lim_n W_{n \wedge t}$ equal to?

Exercises 3

1. Solve each of the equations;

$$B_t = B_0 + \int_0^t r B_s \, ds$$

$$S_t = S_0 + \int_0^t \mu S_s \, ds + \int_0^t \sigma S_s \, dW_s$$

(apply Itô's Lemma to the second equation)

where μ and σ are positive constants.

Solve also,

$$S_t = S_0 + \int_0^t \mu(s) S_s \, ds + \int_0^t \sigma(s) S_s \, dW_s$$

where μ and σ are continuous functions into $[0, \infty)$. Show that this case is "no more general" than the case μ, σ are constants.

2.
(a) Consider a one period (binomial) market where S and B mean the usual things and the time set is $\{0, T\}$. Suppose

$$S_T = \begin{cases} uS_0 & \text{with probability } p \\ dS_0 & \text{" \quad " \quad } 1-p \end{cases}$$

where $u > 1$, $0 < d < 1$, and

$$B_T = (1+r)B_0 \quad , \quad r > 0.$$

Show that there is arbitrage in this model if $r < d$ or $r > u$. You can assume that it's a frictionless market, short selling is allowed, etc.

(13) Imagine now that one has a multi-period market with time set,

$$\{0, t_1, t_2, \ldots, t_n\}$$

and the dynamics of S and B are given by;

$$B_{t_j} = (1+r)^j B_0 \qquad j = 0, 1, \ldots, n$$

$$S_{t_k} = S_0 \prod_{j=1}^{k} (u I^j_{\{+\}} + d I^j_{\{-\}})$$

and the random variables, $u I^j_{\{+\}} + d I^j_{\{-\}}$, $1 \leq j \leq n$, are i.i.d with $\mathbb{P}\{+\} = p$, $\mathbb{P}\{-\} = 1-p$. (So 'this' is the usual binary tree ....). First of all, return to the single period market over the time interval $\{0, t_1\}$. Let $F(S_{t_1})$ be any (bounded) function of S at time t_1. Prove that there is a portfolio of stock and bond which can be adopted at time 0 which is equal to $F(S_{t_1})$ at time t_1. Deduce that if now F is any bounded function of S_{t_n} then there is a portfolio of S and B which can be established at time 0 and then managed in a self-financing manner until time t_n when it will be exactly $F(S_{t_n})$. Argue that the time 0 value $F(S_{t_n})$ must be equal to the time 0 value of this portfolio.

3. A <u>fair</u> coin is tossed repeatedly. We model this by encoding $H=1$, $T=0$, and any sequence of tosses by a binary sequence of 0's and 1's. Regard each toss of the coin as the outcome of an 'investment' where a H means you gain £1 and T means you lose £1. You can adopt as many 'investments' as you wish before any toss of the coin; so, for example, you could 'bet' 5 investments and gain £5 or, worse, lose £5.

Some mathematics: Define Ω, the space of all possible outcomes, as all sequences of 0's and 1's. Assuming the outcomes of each toss of the coin are independent of one another show that

(i) If $E_k = \{\omega = (\omega_i) : \omega_k = 1\}$, $\mathbb{P}(E_k) = \frac{1}{2}$.

(ii) $\mathbb{P}(\{\omega\}) = 0$ for each $\omega \in \Omega$.

Now suppose that you adopt the following strategy: you bet £1 at the first toss. If you win, you stop. If not, you bet £2 at the second toss. If you win, you stop. If not, you bet £4 at the third toss....... if at the k-th toss you win, you stop. If not, you bet £2^k at the $(k+1)^{th}$ toss..... . Show that with probability one this strategy leads to winning £1, albeit requiring very

very deep pockets. As a sobering exercise, calculate the expected gain from this strategy for up to 4 tosses of the coin. (This is easy, just count!).

4. Let S be the 'dollar' value of an asset in the US economy with S a log-normal

$$S_t = S_0 e^{\sigma W_t^S + (\mu_S - \sigma_S^2/2)t}.$$

Let (γ_t) be the exchange rate so that $S_t\gamma_t$ gives the sterling value of the asset S at time t, (so γ_t of sterling is one dollar at time t'). Suppose that,

$$\gamma_t = \gamma_0 + \int_0^t \mu_\gamma \gamma_s \, ds + \int_0^t \sigma_\gamma \gamma_s \, dW_s^\gamma$$

and that $\langle W^S, W^\gamma \rangle_t = \rho t$, where ρ is the correlation coefficient between W^S and W^γ. Use the multiplication rule to show that

$$\gamma_t S_t = \gamma_0 S_0 + \int_0^t (\mu_\gamma + \mu_S) S_s \gamma_s \, ds + \int_0^t \sigma_\gamma \gamma_s S_s \, dW_s^\gamma + \int_0^t \sigma_s \gamma_s S_s \, dW_s^S + \int_0^t \sigma_\gamma \sigma_s S_s \gamma_s \rho \, ds.$$

5. Let $C(t,x) = x N(d_1(T-t,x)) - K e^{-r(T-t)} N(d_2(T-t,x))$

where $d_1 = \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{x}{K}\right) + (r + \sigma^2/2)(T-t)\right)$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, dz .$$

Prove that,

(i) $K e^{-r(T-t)} N'(d_2) = x N'(d_1)$ here N' is the derivative of N........ .

(ii) $\frac{\partial C}{\partial x} = N(d_1)$

(iii) $\frac{\partial C}{\partial t} = -rK e^{-r(T-t)} N(d_2) - \frac{\sigma x}{2\sqrt{T-t}} N'(d_1)$

(iv) Put this together to show that $C(t,x)$ satisfies

$$r\, C(t,x) = \frac{\partial}{\partial t} C(t,x) + r x \frac{\partial}{\partial x} C(t,x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} C(t,x)$$

for $0 \leq t < T$, $x > 0$.

So $C(t,x)$ is a solution of the B+S equation.

What is $\lim_{t \to T} C(t,x)$?

What is $\lim_{x \to \infty} \frac{C(t,x)}{x}$?

What is $\lim_{x \to 0} C(t,x)$?

I leave the final conclusion to you.

6. Recall the discussion of self-financing portfolios: A portfolio, (ϕ, ψ), is

self-financing iff

$$\tilde{V}_t(\phi,\psi) = \tilde{V}_0(\phi,\psi) + \int_0^t \phi_s \, d\tilde{S}_s$$

where "~" denotes discounting. Suppose there is a measure, $\mathbb{Q}$, which is equivalent to $\mathbb{P}$ — the original measure we began with — and such that $\tilde{S}$ is a martingale under $\mathbb{Q}$. Prove that there is no arbitrage between S and B. The point of this exercise is this; $\tilde{S}$ might NOT be a martingale under $\mathbb{P}$, but if there is a measure such as $\mathbb{Q}$, then the argument used in Lectures still carries through.

7. Suppose X is a claim, $X \geq 0$ and (ϕ,ψ) is an admissible strategy which replicates X. Prove that $\phi_t S_t + \psi_t B_t \geq 0$ for $t \in [0,T]$.

8. Suppose X is a claim, $X \geq 0$. Let (ϕ,ψ) and (ϕ',ψ') be a pair of admissible strategies with

$$\phi_T S_T + \psi_T B_T = \phi'_T S_T + \psi'_T B_T = X$$

Make an argument to show that $\phi = \phi'$, $\psi = \psi'$; here 'equals' means "$\mathbb{P}$-almost every path is identical".

9. A call option is written on the square of S. So the payoff is $(S_T^2 - K)^+$. Use the martingale method for pricing to

find the time $t=0$ price of this security.

10. Prove the Call-Put Parity Formula. Use it to give an explicit formula for a Put Option on S struck at K. (If you don't quite get the jargon look in the books or ask around...). Assume that $\mathbb{P}$ is the risk-neutral measure and you might like to start with;

$$(S_T-K)^+ - (K-S_T)^+ = (S_T-K)^+ - (S_T-K)^-$$

recall $f^+ = \max\{f,0\}$, $f^- = \max\{-f,0\}$.

11. Look back at your notes covering the derivation of the Black-Scholes equation. Note that we did not prove that the portfolio is S and B, (ϕ,ψ) where

$$\phi_t(\omega) = \frac{\partial C}{\partial x}(t,S_t)$$

$$\psi_t(\omega) = \frac{\frac{\partial C}{\partial t}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2(\omega)\frac{\partial^2 C}{\partial x^2}(t,S_t)}{r B_t}$$

was self-financing. Finish the proof, that is, show that

$$\phi_t S_t + \psi_t B_t = \phi_0 S_0 + \psi_0 B_0 + \int_0^t \phi_s \, dS_s + \int_0^t \psi_s \, dB_s .$$

12. With the usual S and B, prove that a portfolio, (ϕ,ψ), in S and B respectively, is self-financing iff,

$$\tilde{V}_t(\phi,\psi) = \tilde{V}_0(\phi,\psi) + \int_0^t \phi_s d\tilde{S}_s .$$

Here "$\sim$" denotes a discounted quantity.

13. Suppose that you have a 'stock strategy', ϕ, and you want to make a self-financing portfolio in S and B to implement this stock strategy. How do you choose ψ, the bond strategy? Answers like "carefully" have missed the point of this question.

14. On the filtration generated by a Brownian Motion, W, which has been augmented ... etc ... so that "everything is nice" ..... there live two stocks and a bond. For $i=1,2$,

$$S_t^i = S_0^i + \int_0^t \mu^i S_s^i ds + \int_0^t \sigma^i S_s^i dW_s,$$

$$B_t = e^{rt}.$$

Where, $\mu^i > 0$, $\sigma^i > 0$ are constants. Prove that for each $i \in \{1,2\}$ there is a probability measure $\mathbb{Q}^i$, equivalent to the original measure, $\mathbb{P}$ (under which W is a Brownian Motion), and such that $\tilde{S}^i$ is a martingale under $\mathbb{Q}^i$. Write down the dynamics of the other asset under $\mathbb{Q}^i$. Is it possible for $\mathbb{Q}^1 = \mathbb{Q}^2$? How?

15. Let $(\rho(s))$ be an adapted continuous process such that,

$$-1 \leq \rho(s,\omega) \leq +1 \quad , s \in [0,\infty), \omega \in \Omega.$$

Let B^1 and B^2 be independent Brownian Motions. Set

$$W_t = \int_0^t \rho(s)\,dB_s^1 + \int_0^t (1-\rho(s)^2)^{1/2}\,dB_s^2$$

Is (W_t) a martingale? What is $(\langle W \rangle_t)$? What is the correlation between W and B^1 at time t? Do you have any observations to make about W?

16 Recall our discussion of dividend paying assets: We took a portfolio in S and B with initial value v_0, say, and wrote

$$V_t = v_0 + \int_0^t (V_s - \phi_s S_s) r\,ds + \int_0^t (\mu+q)\phi_s S_s\,ds$$

$$+ \int_0^t \phi_s S_s \sigma\,dW_s$$

after a little rearrangement we get

$$\tilde{V}_t = v_0 + \int_0^t (\mu+q-r)\,\tilde{S}_s \phi_s\,ds + \int_0^t \phi \tilde{S}_s \sigma\,dW_s,$$

enter Girsanov and the measure $\mathbb{Q}_q$ under which, $W_t^q = W_t + \frac{(\mu+q-r)t}{\sigma}$ is a BM.

So $\tilde{V}_t = v_0 + \int_0^t \phi_s \tilde{S}_s \sigma\,dW_s^q$. Our discounted portfolio is a martingale under $\mathbb{Q}_q$ and whatever random variable V_T is, we know that it's time $t=0$ value must be

$$\mathbb{E}^{\mathbb{Q}_q}(\tilde{V}_T).$$

We saw also that the tacit assumption that the existence of dividends amounts to the rate of return of S being $\mu + q$, led to (financial) nonsense.

So, what are the dynamics of S? We have assumed that they are,

$$S_t = S_0 e^{\sigma W_t + (\mu - \sigma^2/2)t}.$$

This being so the following argument holds: Consider a portfolio with no bond. Then all dividends are immediately invested back in S ('taken as scrip'?). Now V_t satisfies

$$V_t = v_0 + \int_0^t S_s \phi_s (\mu + q) ds + \int_0^t V_s \sigma dW_s$$

but $V_s = \phi_s S_s$ so

$$V_t = v_0 + \int_0^t V_s (\mu + q) ds + \int_0^t V_s \sigma dW_s$$

i.e.

$$V_t = v_0 e^{\sigma W_t + (\mu + q - \sigma^2/2)t}$$

$$= v_0 e^{qt} S_t$$

So, at time T, if $v_0 = S_0$ then $V_T = e^{qT} S_T$, and the time $t = 0$ value of this portfolio is

$$\mathbb{E}^{Q^q}(\tilde{V}_T) = \mathbb{E}^{Q^q}(e^{-(r-q)T} S_T).$$

as

$$S_T = S_0 e^{\sigma W_T + (\mu - \sigma^2/2)T} = S_0 e^{\sigma\left(W_T + \left(\frac{\mu + q - r}{\sigma}\right)T\right) + ((r-q) - \sigma^2/2)T}$$

$$= S_0 e^{\sigma W_T^q + ((r-q) - \sigma^2/2)T}.$$

and $\quad e^{-(r-q)T} S_T = S_0 e^{\sigma W_T^q - \frac{\sigma^2}{2}T}$ so that

$$\mathbb{E}^{\mathbb{Q}^q}\left(e^{-(r-q)T} S_T \right) = S_0 .$$

So an investment of S_0, in S at time 0, yeilds $e^{qT} S_T$ at time T. So an invest--ment of $e^{-qT} S_0$ leads to exactly S_T at time T. Consequently the forward price of S at time T determined at time 0 will be $e^{(r-q)T} S_0$. The hedge is to borrow $e^{-qT} S_0$ and invest this in S at time 0. At time T your assets are S_T and your liabilities $e^{(r-q)T} S_0$ - which you obtain from the counter-party for the delivery of S_T.

Write down the value of the call option on S, strike K and expiry T.

Is $\tilde{S}$ a martingale under $\mathbb{Q}^q$?

Recall,

$$\tilde{V}_t = v_0 + \int_0^t \phi_s \tilde{S}_s \sigma \, dW_s^q$$

is it $$= v_0 + \int_0^t \phi_s \, d\tilde{S}_s \quad ???$$

If you read around this topic you might find the following as a departure point:

"Dividends paid by a stock reduce its value, and so.....

$$S_t = S_0 + \int_0^t \mu S_s \, ds + \int_0^t \sigma S_s \, dW_s - \int_0^t q S_s \, ds \ldots"$$

Enjoy!

17. Let (X_n) be a process adapted to a stochastic base $(\Omega, \mathcal{F}_\infty, \mathbb{P}, (\mathcal{F}_n), \mathbb{N})$, so the time index here is $\mathbb{N}$, the natural numbers. It is always possible to write (X_n) as the sum of a martingale and another process — this much is trivial; let (N_n) be <u>any</u> martingale,

$$X_n = N_n + (X_n - N_n), \quad \forall n.$$

But you can build the martingale directly from the process (X_n): set,

$$U_1 = X_1$$

and $\quad U_2 = X_1 + X_2 - M_1(X_2)$

then $\quad M_1(U_2) = X_1 = U_1$, so, continuing

$$U_{n+1} = U_n + X_{n+1} - M_n(X_{n+1})$$

We can now write (as before)

$$X_n = U_n + (X_n - U_n) = U_n + A_n, \text{ say.}$$

The process, (A_n), has an interesting feature.

$$A_n = X_n - U_n = X_n - (U_{n-1} + X_n - M_{n-1}(X_n))$$
$$= M_{n-1}(X_n) - U_{n-1} \in L^1(\mathcal{F}_{n-1}).$$

So A_n is $\mathcal{F}_{n-1}$ measurable, we say that (A_n) is <u>predictable</u>.

So, any L^1-process can be 'decomposed' into the sum of a martingale and a predictable process. This is called the Doob-Meyer decom-position of a process.

(i) Suppose that (X_n) is a submartingale, that is, $M_n(X_{n+1}) \geq X_n$. Prove that (A_n) in the Doob-Meyer decomposition of (X_n) is a predictable increasing process with $A_1 = 0$.

(ii) Now suppose that (X_n) is a supermartingale; $M_n(X_{n+1}) \leq X_n$. Show that

$$X_n = U_n - A_n$$

where (U_n) is a martingale and (A_n) is a predictable increasing process with $A_1 = 0$.

18. Let σ and τ be stopping times taking on two distinct values,

$$\sigma = s_1 I_{\{\sigma = s_1\}} + s_2 I_{\{\sigma = s_2\}}$$

$$\tau = t_1 I_{\{\tau = t_1\}} + t_2 I_{\{\tau = t_2\}}$$

and suppose that $\sigma(w) \leq \tau(w)$, for $w \in \Omega$.

For $X_\tau \in L^2(\mathcal{F}_\tau)$ define,

$$M_\sigma(X_T) = M_{s_1}(X_T) I_{\{\sigma = s_1\}} + M_{s_2}(X_T) I_{\{\sigma = s_2\}}$$

and $M_\tau(X_T)$ similar. Prove that

$$M_\sigma \circ M_\tau = M_\sigma$$

and for a martingale, $(Y_n) = Y$,

$$(Y^\tau)^\sigma = Y^\sigma .$$

19. Let $(\Omega, \mathcal{F}_T, \mathbb{P}, (\mathcal{F}_n), 0 \leq n \leq T)$ be a discrete time stochastic base (so, $n, T \in \mathbb{N}$). Let (X_n) be an adapted process. Consider the following process defined by a 'backward iteration'.

Write, $Z_T = X_T$.

$$Z_{T-1} = \max\{X_{T-1}, M_{T-1}(X_T)\}$$

$$Z_{T-2} = \max\{X_{T-2}, M_{T-2}(Z_{T-1})\}$$

$$\vdots$$

$$Z_{T-k} = \max\{X_{T-k}, M_{T-k}(Z_{T-k+1})\}$$

$$\vdots$$

until Z_1.

Prove that $Z_k \geq X_k$ and $Z_k \geq M_k(Z_{k+1})$ for $1 \leq k \leq T-1$. So that "(Z_k) is a supermartingale dominating (X_k)".

Define a stopping time:

$$\tau^*(\omega) = \min\{n : X_n = Z_n\}$$

Prove that:

(i) $\forall \omega \in \Omega \quad \tau^*(\omega) \leq T$

(ii) $X_{\tau^*} = Z_{\tau^*}$

(iii) This part we take step by step.

(α) We are interested in the process $Z^{\tau^*} \equiv (Z_{\tau^* \wedge n})$. Refer back to earlier exercises;

$$Z_{\tau^* \wedge (n+1)} = \sum_{\ell \leq n} Z_\ell I_{E_\ell} + Z_{n+1} I_{\{\tau^* \geq n+1\}}$$

if you didn't prove this earlier, take the time to do it now.

(β) Prove that $\{\tau^* \geq n+1\} \in \mathcal{F}_n$.

(γ) Think about Z_n on $\{\tau^* \geq n+1\}$: refer to the definition Z_n and τ^* to conclude that $Z_n = M_n(Z_{n+1})$ on $\{\tau^* \geq n+1\}$. This is a crucial step make sure you understand it.

(δ) Consider $M_n(Z_{\tau^* \wedge (n+1)})$. Use ($\beta$) first of all to conclude,

$$M_n(Z_{n+1} I_{\{\tau^* \geq n\}}) = M_n(Z_{n+1}) I_{\{\tau^* \geq n+1\}}$$

then use (γ) to modify M_n (the right hand side) in (α) above.

(ϵ) Prove that $(Z_{\tau^* \wedge n})$ is a martingale.

20. Suppose we have the usual "one dimensional" model with a cash account, $B = e^{rt}$, a log-normal stock, S, driven by a standard^t Brownian Motion and a risk-neutral measure $\mathbb{P}$ such that all assets discounted by B are $\mathbb{P}$ martingales. Suppose also that there is a 'zero-coupon bond', whose value at time $t \in [0,T]$ is $B(t,T)$ and $B(T,T) = 1$. Prove that $B(t,T) = e^{-r(T-t)}$. Use $(B(t,T))$ as a numeraire and write down the measure under which all assets discounted by $(B(t,T))$ are martingales. Take also S, the stock as numeraire. Write down the measure under which all assets discounted by S are martingales. Finally, consider a call option*, C, written on S with strike K and expiry T. Show that the price of C at time t, $\pi_t(C)$, is

$$\pi_t(C) = S_t \, M^S_t \left(I_{\{S_T > K\}} \right) - K B(t,T) M^{B(T,T)}_t \left(I_{\{S_T > K\}} \right)$$

where M^S is the condition expectation with respect to the martingale measure for the numeraire S while $M^{B(T,T)}$ is the conditional expectation for the martingale measure for the numeraire $(B(t,T))$.

* a European.

21. Consider a 'market' with two stocks and a cash account;

$$S^i_t = S^i_0 + \int_0^t \mu^i S^i_s \, ds + \int_0^t \sigma^i S^i_s \, dW_s$$

for $i = 1, 2$, and $B_t = e^{rt}$. Notice that each stock is driven by the same Brownian Motion.

The discounted stock price $\tilde{S}^i_t$ can be written

$$\tilde{S}^i_t = \tilde{S}^i_0 + \int_0^t \tilde{S}^i_s \sigma^i d\left(W_s + \left(\frac{\mu^i - r}{\sigma^i}\right)s\right)$$

For each discounted stock to be a martingale under some measure, $\tilde{\mathbb{P}}$, we would need each of $\left(W_t + \left(\frac{\mu^i - r}{\sigma^i}\right)t\right)$, $i = 1, 2$, to be martingales under $\tilde{\mathbb{P}}$ and would hope that they might be Brownian Motions. Can you suggest a condition under which each of these processes are Brownian Motions?

Consider a portfolio comprising ϕ^1_t of S^1_t, ϕ^2_t of S^2_t, ψ_t of B_t. Write $V_t = \phi^1_t S^1_t + \phi^2_t S^2_t + \psi_t B_t$. Prove that the portfolio is self-financing iff

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \phi^1_s d\tilde{S}^1_s + \int_0^t \phi^2_s d\tilde{S}^2_s .$$

Suppose that $\mu^1 - r/\sigma^1 > \mu^2 - r/\sigma^2$ and consider a portfolio comprising $1/S^1_t\sigma^1$ units of S^1_t and $-1/S^2_t\sigma^2$ units of S^2_t, borrowing, or investing, cash to support this position. At time zero the setup cost is $1/\sigma^1 - 1/\sigma^2$. We borrow cash for this cost if it is positive and we invest in cash if this is negative, so the time zero cost of the portfolio, $V_0 = 0$.

Assume this portfolio is self-financing. Prove that

$$\tilde{V}_t = \int_0^t \left(\left(\frac{\mu^1 - r}{\sigma^1}\right) - \left(\frac{\mu^2 - r}{\sigma^2}\right)\right) e^{-rt} dt$$

deduce that there is an arbitrage in this market.

22. Let S be a stock, B a bond and $\mathbb{P}^1$ and $\mathbb{P}^2$ be measures which are "risk-neutral for B". Suppose also that the market consisting of S and B is complete i.e. every L^2 contingent claim can be replicated by a self-financing admissible portfolio in S and B. Assuming S is adapted to the filtration $(\Omega, \mathcal{F}_T, \mathbb{P}, (\mathcal{F}_t), [0,T])$ and $\mathbb{P}^1$ and $\mathbb{P}^2$ are defined on $\mathcal{F}_T$, prove that $\mathbb{P}^1 = \mathbb{P}^2$.

Hint: let $E \in \mathcal{F}_T$ and consider the claim I_E.